Quotient Spaces (or Identification Spaces)

Definition Let X and Y be topological spaces; let $p : X \to Y$ be a surjective function. The map p is said to be a quotient map, provided a subset U of Y is open in Y if and only if $p^{-1}(U)$ is open in X.

Remark

- This condition is stronger than continuity.
- An equivalent condition is to require that a subset A of Y is closed in Y if and only if $p^{-1}(A)$ is closed in X. Equivalence of the two conditions follows from the equation

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B).$$

- A subset C of X is saturated (with respect to the surjective continuous function $p : X \to Y$) if C contains every set $p^{-1}(y)$ that it intersects. Thus a subset C of X is saturated if $C = p^{-1}(p(C))$.

  So, p is a quotient map if p is continuous and p maps saturated open sets of X to open sets of Y (or saturated closed sets of X to open sets of Y).
- Recall that a map $f : X \to Y$ is said to be an open map if for each open set U of X, the set $f(U)$ is open in Y. It is said to be an closed map if for each closed set A of X, the set $f(A)$ is closed in Y.

  It follows immediately from the definition that if $p : X \to Y$ is a surjective continuous map that is either open or closed, then p is a quotient map.

Definition If X is a space and A is a set and if $p : X \to A$ is a surjective function, then there exists exactly one topology $\mathscr{T}$ on A relative to which p is a quotient map; it is called the quotient topology induced by p.

Remark Since $\mathscr{T}$ is defined by

$$\mathscr{T} = \{U \subseteq A \mid p^{-1}(U) \text{ is open in } X\},$$

and since

$$p^{-1}(\emptyset) = \emptyset \text{ and } p^{-1}(A) = X \implies \emptyset,\, A \in \mathscr{T}$$

$$p^{-1}\left(\bigcup_{\alpha\in J} U_\alpha\right) = \bigcup_{\alpha\in J} p^{-1}(U_\alpha) \implies \text{ if } U_\alpha \in \mathscr{T},\, \forall\, \alpha \in J \text{ then } \bigcup_{\alpha\in J} U_\alpha \in \mathscr{T}$$

$$p^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \bigcap_{i=1}^{n} p^{-1}(U_i) \implies \text{ if } U_i \in \mathscr{T},\, \forall\, 1 \le i \le n \text{ then } \bigcap_{i=1}^{n} U_i \in \mathscr{T}$$

$\mathscr{T}$ is a topology.

Examples

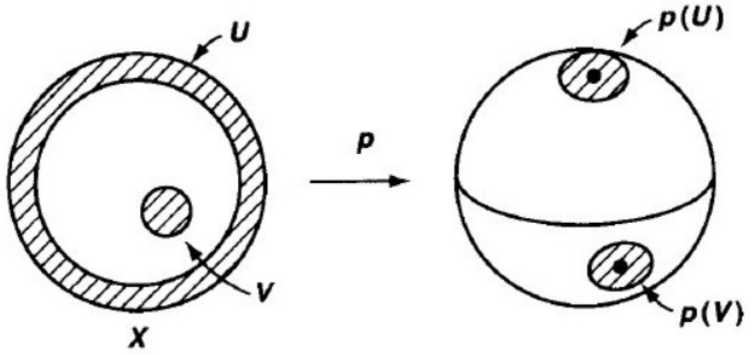
1. Let $\pi_1 : X \times Y \to X$ be the projection map; π_1 is continuous and surjective. If $U \times V$ is a basis element for $X \times Y$, the image set $\pi_1(U \times V) = U$ is open in X. It follows readily that π_1 is an open map. In general, π_1 is not a closed map; the projection $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ carries the closed set $\{(x, y) \mid xy = 1\}$ onto the the nonclosed set $\mathbb{R} \setminus \{0\}$, for instance.

2. Let X be the subspace $[0,1] \cup [2,3]$ of $\mathbb{R}$, and let Y be the subspace $[0,2]$ of $\mathbb{R}$. The map $p : X \to Y$ defined by
$$p(x) = \begin{cases} x & \text{for } x \in [0,1], \\ x-1 & \text{for } x \in [2,3] \end{cases}$$
is readily seen to be surjective, continuous, closed map. It is not, however, an open map; the image of the open set $[0,1]$ of X is not open in Y.
3. Let $A = \{a, b, c\}$ be a set of three points and let $p : \mathbb{R} \to A$ be defined by
$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0 \end{cases}$$
Then the quotient topology on A induced by π is $\mathscr{T} = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset\}$, p is an open map and it is not a closed map.

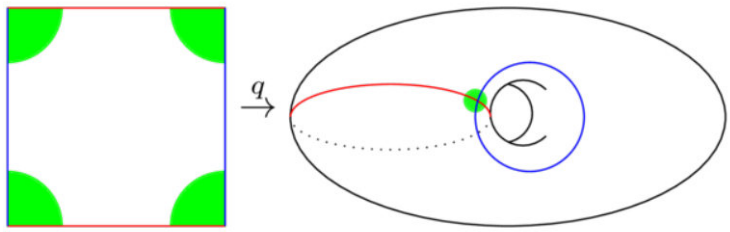
Definition Let X be a topological space, and let X^* be a partition of X into disjoint subsets whose union is X. Let $p : X \to X^*$ be the surjective map that carries each point of X to the element of X^* containing it. In the quotient topology induced by p, the space X^* is called a quotient space (or an identification space) of X.

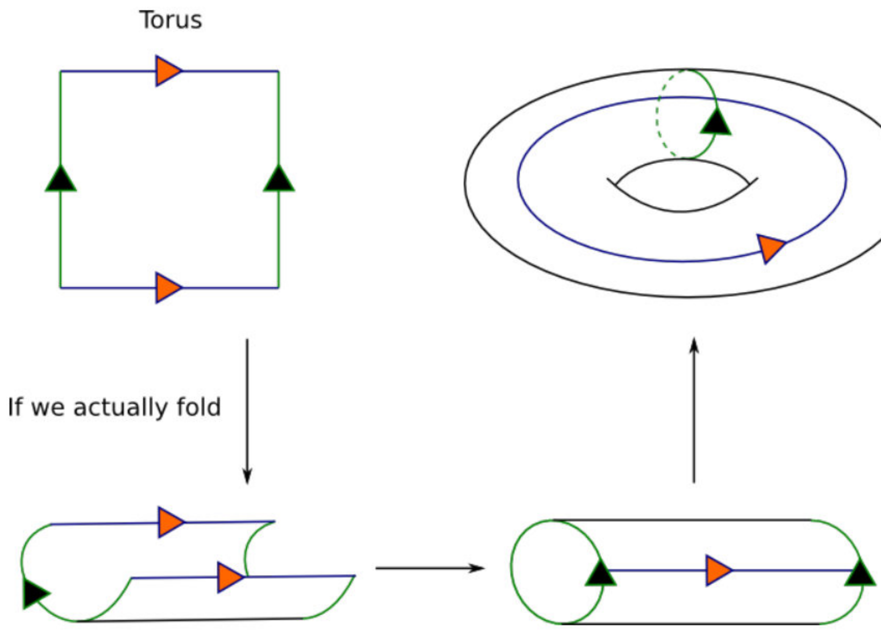
Examples

4. Let X be the closed unit ball $\{(x,y) \mid x^2+y^2 \leq 1\}$ in $\mathbb{R}^2$, and let X^* be the partition of X defined as follows:
$$X^* = \{\, X = \{(x,y)\}_{x^2+y^2<1},\ \{(x,y) \mid x^2+y^2 = 1\}\,\}.$$
Note that each $\{(x,y)\} \in X^*$ is a one-point subset of X while $\{(x,y) \mid x^2+y^2 = 1\}$



is a circle subset of X. Typical open sets in X of the form $p^{-1}(U)$ are pictured by the shaded regions in the following figure. One can show that X^* is homeomorphic with the unit 2-sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$.
5. Let X be the rectangle $[0,1] \times [0,1]$ in $\mathbb{R}^2$. Define a partition X^* of X as follows:
$$X^* = \{\, \{(x,y)\}_{0<x,\,y<1},\ \{(x,0),\ (x,1)\}_{0<x<1},\ \{(0,y),\ (1,y)\}_{0<y<1},\ \{(0,0),\ (0,1),\ (1,0),\ (1,1)\}\,\}.$$
There are 4 kinds of typical open sets in X of the form $p^{-1}(U)$ and one of them is shown in the following figure.
This description of X^* is just a mathematical way of pasting the edges of a rectangle together to form a torus.

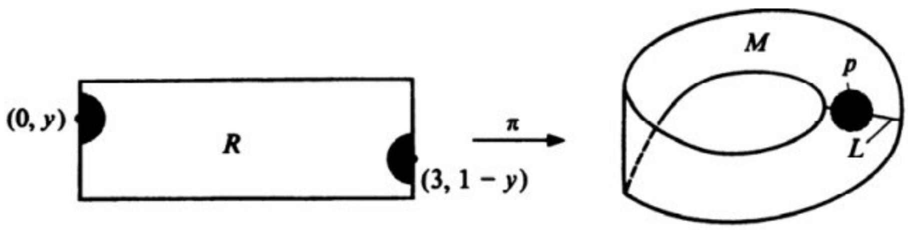




6. Let R be the rectangle $[0,3] \times [0,1]$ in $\mathbb{R}^2$. Define a partition R^* of R as follows:

$$R^* = \{\, \{(x,y)\}_{0<x<3,\, 0\leq y\leq 1},\ \{(0,y),\ (3,1-y)\}_{0\leq y\leq 1}\,\}.$$

Identify the subsets of R^* with the points of our Möbius strip M, and define the map $\pi : R \to M$ by sending each point of R to the subset of the partition in which it lies.



Note that the union of two half discs in R, centres $(0,y)$, $(3,1-y)$ and of equal radius, maps via π to an open neighborhood of p in the identification topology on M, and if we take a single half-disc, its image in M is not a neighborhood of p and is not open, so π is not an open mapping.

7. Consider the subspace $A = [0,1] \cup (2,3]$ of $\mathbb{R}$; it is a subspace of the space $X = [0,1] \cup [2,3]$ of Example 2. Suppose that we restrict the map p of Example 2 to A. Then $q = p|_A : A \to [0,2]$

is continuous and surjective, but it is not a quotient map. The set

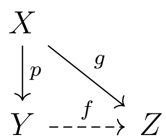
$$q^{-1}((1,2]) = (2,3],$$

for instance, is closed in the domain space A; but $(1,2]$ is not closed in the image space $Y = [0,2]$.

So, if $p : X \to Y$ is a quotient map and A is a subspace of X, then the map $p' : A \to p(A)$ obtained by restricting both the domain and range of p need not be a quotient map. However, it is easy to see that if A is open in X and p is an open map, then ($p' = p|_A$ is an open map and) p' is a quotient map; the same is true if both A and p are closed.

8. Let Y be the subspace $(\overline{\mathbb{R}}_+ \times \mathbb{R}) \cup (\mathbb{R} \times 0)$ of $\mathbb{R} \times \mathbb{R}$; let $h = \pi_1|_Y$. For any subset U of $\mathbb{R}$, since $h^{-1}(U) \cap (\mathbb{R} \times 0) = U \times 0$, h is a quotient map while h is neither open nor closed. For instance, the set $A = \{(x,y) \mid x^2 + (y-2)^2 < 1, x \geq 0\}$ is open and the set $B = \{(x,y) \mid xy = 1, x > 0\}$ is closed in Y, while $h(A) = [0,1)$ is not open and $h(B) = (0,\infty)$ is not closed in $\mathbb{R}$.

Theorem Let $p : X \to Y$ be a quotient map. Let Z be a space and let $g : X \to Z$ be a continuous function that is constant on each set $p^{-1}(\{y\})$, for $y \in Y$. Then g induces a continuous function $f : Y \to Z$ such that $f \circ p = g$.



Proof For each $y \in Y$, since g is constant on $p^{-1}(\{y\})$, the set $g(p^{-1}(\{y\}))$ is a one-point set in Z and $f(y)$ can be defined as

$$f(y) = g(p^{-1}(\{y\})).$$

So, we have defined a map $f : Y \to Z$ such that

$$f(p(x)) = g(x) \quad \text{for each } x \in X.$$

Given an open set V of Z, since g is continuous and p is a quotient map,

$$g^{-1}(V) = p^{-1}(f^{-1}(V))$$

is open in X and $f^{-1}(V)$ is open in Y. This shows that f is continuous.

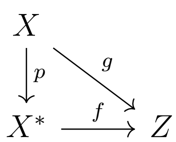
Theorem Let $g : X \to Z$ be a surjective continuous function. Let X^* be the following collection of subsets of X:

$$X^* = \{g^{-1}(\{z\}) \mid z \in Z\}.$$

Give X^* the quotient topology.

(a) If Z is Hausdorff, then so is X^*.

(b) The map g induces a bijective continuous function $f : X^* \to Z$, which is a homeomorphism if and only if g is a quotient map.



Proof By the preceding theorem, g induces a continuous function $f : X^* \to Z$; it is clear that f is bijective. If Z is Hausdorff, then given distinct points of X^*, their images under f are distinct and thus possess disjoint neighborhoods U and V. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint neighborhoods of the two given points of X^*.

Suppose that f is a homeomorphism. Then both f and the projection $p : X \to X^*$ are quotient maps, so that $g = f \circ p$ is a quotient map.

Conversely, suppose that g is a quotient map. Given an open set V of X^*, since p is continuous,

$$g^{-1}(f(V)) = p^{-1}(V)$$

is open in X and $f(V)$ is open in Z, because g is a quotient map. Hence f maps open sets to open sets, so it is a homeomorphism.

Theorem Let $f : X \to Y$ be an onto continuous function. If f maps open sets of X to open sets of Y, or closed sets to closed sets, then f is a quotient map.

Proof Suppose f maps open sets to open sets.

If U is open in Y, since f is continuous, $f^{-1}(U)$ is open in X.

Conversely, if $f^{-1}(U)$ is open in X for some subset U of Y, since f is onto and f is an open mapping, $U = f(f^{-1}(U))$ is open in Y. Hence f is a quotient map.

Corollary Let $f : X \to Y$ be an onto continuous function. If X is compact and Y is Hausdorff, then f is a quotient map.

Proof Let C be a closed subset of X. Since X is compact, $f : X \to Y$ is continuous and Y is Hausdorff, $f(C)$ is a compact subset and hence a closed subset of Y. This implies that f takes closed sets to closed sets. Hence f is a quotient map.